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Extended summary

Extended analysis of the 3-cpu reconfigurable class of parallel robotic manipulators

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Abstract. During past years, two different minor mobility parallel robots have been developed and prototyped at the Laboratory of Robotics of the Polytechnic University of Marche in Ancona: the first one is a pure translational machine, called I.Ca.Ro. [1], characterised by a Cartesian achitecture driven by linear modules while the second one, Sphe.I.Ro. [2], provides the mobile platform with a spherical motion, driven by 3 linear induction motors. Notwithstanding the quite different kinematic performances of the machines [3], they are both based on the same 3-CPU architecture, with of course a different setting of the joints. The present work was aimed at investigating whether a common mechanical architecture would be able to provide both motions by a simple reconfiguration or even if the same machine could yield the two different kinds of motions by meeting some "switching configuration", i.e. whether it could show a kinematotropic behaviour [4].

This thesis also presents a workspace analysis of the spherical 3-CPU parallel machine. The spherical wrist here investigated represents a particular operational mode of the kinematotropic 3-CPU parallel robot studied in the first section. The spherical working mode represents a rele-

vant case among 3-CPU motion capabilities since it shares common configurations with all other working modes. Because of that, its presence effectively allows the switch between all kinds of mobilities offered by the machine. It is known that transitions between two operational behaviours take place on singular configurations, i.e. when robot limbs can move independently from platform displacement/rotation and assume a configuration that allows a different kind of motion. Therefore, the sub-spaces of actuated joints space allowing such transitions must represent singularity surfaces of the manipulator [5,6] and they should match the singularity regions detection carried on by means of Jacobian matrix analysis [7]. Thus, results achieved in the first section are compared with the workspace analysis performed through the study of the manipulator Jacobian matrix. To do this, the forward position kinematics of the manipulator is briefly faced to provide the solution which is needed for the formulation of the differential kinematics. Theory of screws is then exploited for the formalization of velocity kinematics problem.

Keywords. parallel kinematics machines, kinematotropic mechanisms, recoconfigurable mechanisms, algebraic geometry.

1 Problem statement and objectives

The key idea was to realize a reconfigurable universal joint that allows to vary robot kinematics without changes in legs structure. A simple solution to this problem is using a spherical joint made of three consecutive revolute pairs (see figure 1(a)): in this way, three different universal joints are obtained by locking, one at a time, the three rotations of the spherical joint. It is worth to remark that both I.Ca.Ro. and Sphe.I.Ro. are comprehended within these joints configurations. The extended analysis carried on in pointed out that the only chance to yield different kinds of motions with the 3-CPU configuration is to lock the first rotation of spherical joint, thus obtaining the kinematics scheme of the robot Sphe.I.Ro. which, according to Karouia and Hervé [8], is able to provide spherical motions: the universal joint is made of two revolute pairs, the first one perpendicular to the plane of the leg and the second lying on leg plain intersecting centre of platform reference frame.

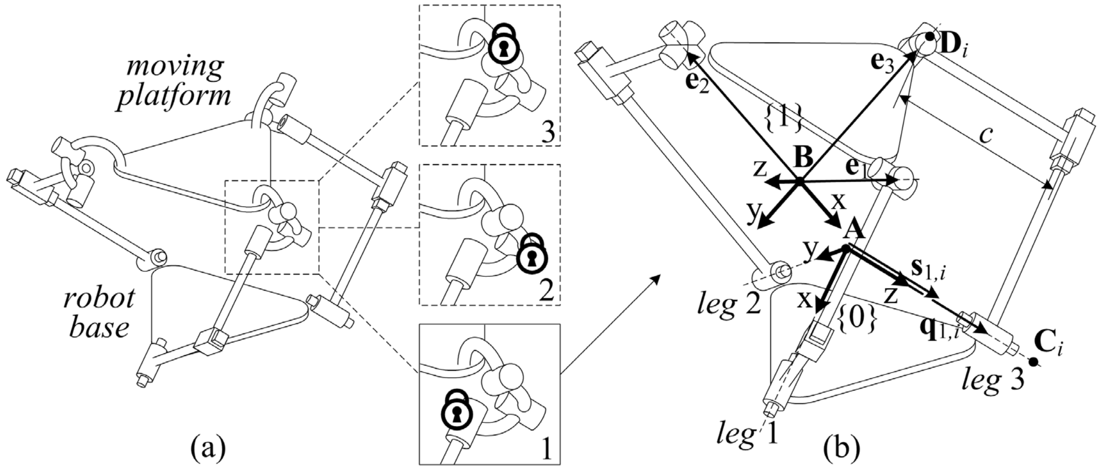

Figure 1: possible arrangement of the 3 revolute joints equaivalent to a reconfigurable spherical joint (a) and 3-CPU architecture able to yield different motions (b).

The study of mechanism kinematics requires to use a parametrization possibly free of representation singularities. To this aim the transformation matrix ${}^{0}\mathbf{T}_1$ between the two frames {0} and {1} (see figure 1.b) is expressed as a function of Study's parameters, which are 8 parameters x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3 which define a point in the 6-dimensional quadric S in the projective space P^7, a semi-algebraic set defined by [9,10]:

$$\begin{aligned} & x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3 = 0 \\ & x_0^2 + x_1^2 + x_2^2 + x_3^2 \neq 0 \end{aligned} \tag{1}$$

In order to obtain a complete algebraic description of robot kinematics, Study quadric equations (1), that are intrinsic of the used parametrization, must be juxtaposed to relations characteristic of the specific legs architecture. Among these constraints equations, a distinction can be made based on their dependence on robot actuation. Indeed, each leg is made of a serial kinematic chain whose joints constrain the manipulator mobility regardless of the actuation parameters. On the other hand some equations are needed to describe the influence of actuation displacements on end effector pose. In the following a geometric interpretation of legs mobility will express actuation independent constraints while loop closure equations will provide actuation influence [11]. The robotic system is then fully algebraically described by a set of polynomial constraint equations, that can be collected in the polynomial ideal

$$\mathcal{I} = \left\langle g_1, g_2, g_3, g_{4,A}, g_{5,A}, g_{6,A}, g_7, g_8 \right\rangle \tag{2}$$

As suggested by Walter et al. [12], a simpler formulation of the constraint equations $\mathcal{I}$ is fundamental for a deeper understanding of robot kinematic behaviour. With this aim, the authors developed a specialized study on that portion of the polynomial ideal collecting those constraints equations purely dependent on kinematic architecture. Therefore, the sub-ideal is analysed through the computation of its primary decomposition; this method consists of splitting the ideal into several sub-ideals such that the union of their vanishing sets correspond to the vanishing set of the starting ideal. It is worth to remark that the zero set, or vanishing set, of a polynomial ideal is the set of all points that simultaneously satisfy the homogeneous equations composing the ideal.
The vanishing set of the polynomial ideal (2) also yield a complete mathematical descritption of of robot forward kinematics problem (FKP) whose solution can be carried on by means of algebraic geometry tools [13-16] able to reduce the complexity of the problem, i.e. the computation of the Groebner [17,18] basis of the ideal. In the case of a pure rotational behavior, whose existence is proved by the computation of the primary decomposition of (2), observation of the basis reveals that on of the 4 generators is a univariate polynomial of degree 16 in the unknown Study's parameter x_3. If equalled to zero, roots of such polynomial yields to 16 values of x_3 that correspond to many solutions of the FKP. Indeed, the zero set of the other basis polynomials constitute a linear system of equations in x_0, x_1 and x_2 if roots of x3 are substituted one at a time. Therefore, 16 feasible sets of Study's parameters are reachable for a single actuation displacements set. Notwithstanding the number of feasible parameters sets, the actual number of configurations performed by robotic manipulator is 8 as suggested for this type of manipulators by Innocenti and Parenti-Castelli in [19]. Indeed, observation of numerical examples carried on such kinematics model revealed that the constraint equations here proposed produce sets coincident in pairs: consequently only 8 of them are valid being the others just repeated solutions.
The knowledge of the solutions of the FKP is necessary fo the efficient formulation of PKM's differential kinematics, which is a relevant issue of robot kinematics. The direct differentiation of position equations often requires a computational effort that can be avoided by using different methods for formalization of differential kinematics. In this work, the manipulator Jacobian matrix has been found by means of screw theory with a minimal amount of symbolic computation.
The computation of end-effector Jacobians does not require an explicit knowledge of non actuated joints screws [20-23]. What is needed is the expression of the 3 unit screws of actuated joints together with that of the reciprocal screw of passive joints also present in each leg:

$$\begin{bmatrix} \hat{\$}_{r,1}^T \\ \hat{\$}_{r,2}^T \\ \hat{\$}_{r,3}^T \end{bmatrix} \begin{bmatrix} \boldsymbol{\omega} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} \left[\hat{\$}_{r,1}^T \hat{\$}_{1,1}\right] & 0 & 0 \\ 0 & \left[\hat{\$}_{r,2}^T \hat{\$}_{2,1}\right] & 0 \\ 0 & 0 & \left[\hat{\$}_{r,3}^T \hat{\$}_{3,1}\right] \end{bmatrix} \begin{bmatrix} \dot{q}_{1,1} \\ \dot{q}_{2,1} \\ \dot{q}_{3,1} \end{bmatrix} \tag{3}$$

which is the usual velocity mapping in the form $\mathbf{J}_X \dot{\mathbf{x}} = \mathbf{J}_Q \dot{\mathbf{q}}$.

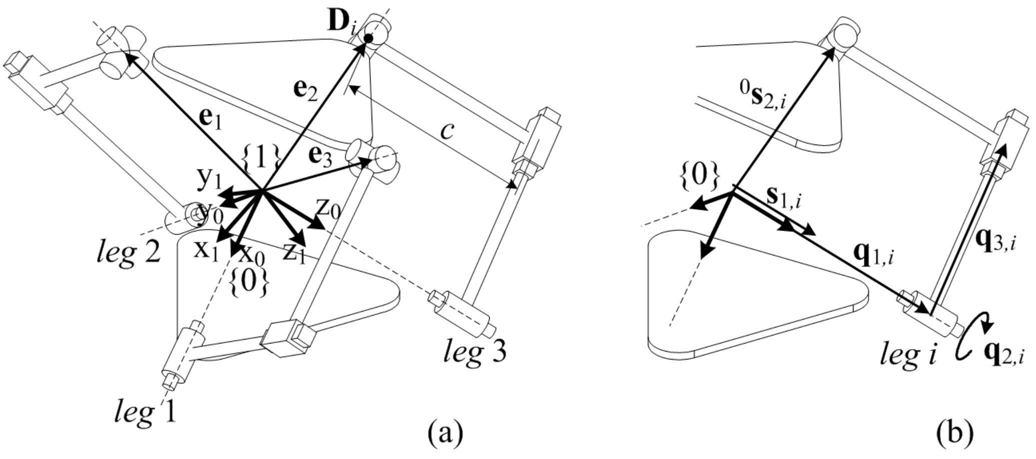

Figure 2: geometric parameters of 3-CPU rotational platform (a) and joints variables and screws directions for i^{th} leg (b).

2 Research planning and activities

As mentioned, each one of the sub-ideals does not complete by its own the characterization of the kinematics problem since its vanishing set just refers to a portion of the feasible solutions of the starting ideal. In particular, sub-ideals denote the different types of mobility that the robotic platform is able to perform. Such conclusion turns evident if homogeneous equations provided that they contain are substituted into end-effector homogeneous transformation matrix, which assumes a different shape for each one of the sub-ideals. A deeper investigation on every sub-ideal revealed that several types of motions are allowed by the 3-CPU architecture. The transformations yielded by each substitution are not analytically reported but a brief description of the different working modes is provided in the following:

- 1 pure rotational working mode, already widely studied by several past works.
- 4 ideals pointed out the same type of mobility, characterized by 3 spurious DOFs: the end-effector can change its orientation rotating about two distinct axes and it can translate among a direction which rotates solidly with the moving platform.
- 6 not usable working modes: the mobilities deriving from these ideals can not be exploited by the actuation chosen for the 3-CPU manipulator, since for each one of them the end-effector is only allowed to move on a plane perpendicular to one of the cylindrical joints passing through the origin of the absolute reference frame {0}. In these configurations, at least one of the cylindrical pairs is not allowed to translate: the mechanical system gets stuck in an under-actuated situation and one of platform DOFs becomes uncontrollable.
- 4 ideals denoted the capability of the ability to move without varying its own configuration with respect to the reference frame, i.e. to perform pure translational motions. The different modes are distinguished for the different orientations of the manipulator, which are fixed in these cases.

This mathematical process allows to demonstrate that it is possible to obtain a **multifunctional robot** by using the same 3-CPU architecture, since it proved capable of both pure rotational and pure translational motions: therefore it is relevant to investigate the possibility to switch from a type of mobility to the other.

The design of a novel manipulator must be preceded by an accurate estimation of the performances the machine is able to provide. To this aim it is worth to carefully analyze the robot workspace, that is in most cases done taking advantage of the information that the differential kinematics matrix provides. Equation (3) provides an explicit expression of the Jacobian matrix as a function of joints passive variables $q_{i,j}$ which are themselves computable in terms of actuated joints displacements $q_{1,1}\ q_{1,2}\ q_{1,3}$. Determinant computation of JX yields:

$$\det \mathbf{J}_X = q_{3,1} q_{3,2} q_{3,3} \left(\cos q_{2,1} \cos q_{2,2} \cos q_{2,3} + \sin q_{2,1} \sin q_{2,2} \sin q_{2,3} \right) \tag{3}$$

By letting equation (3) vanish, singularity surfaces of robot workspace can be detected. The Jacobian matrix determinant can be evaluated for a given set of actuation parameters using the position kinematics.

Analysis of determinant of Jacobian matrix is probably the most used tool for detection of singularity surfaces. Nevertheless, it is often difficult to find the actual mathematical equation describing the borders of robot workspace in terms of actuation variables; in the specific field of PKM, this can represent an impossible task since in most cases the solution of the forward kinematics is not achievable in closed form. In the specific case of the rotational 3-CPU platform, forward kinematics solution yields 8 different configurations for each set of actuators displacements, deriving from zeroing of a polynomial of same degree; therefore, a closed form symbolical solution is actually not achievable and it is not possible to explicitly express equation (3) in terms of actuators displacements. In next section it is shown how the decomposition of constraint polynomials ideal can be exploited for the exact formulation of singularity surfaces that represent the borders of robot workspace.

3 Analysis and discussion of main results

As said, each one of robot working modes is denoted as the vanishing set of one of the 15 significant sub-ideals. In order to be a feasible configuration for a switch between two working modes, a particular pose must belong to both the characteristic vanishing sets. As a matter of fact, such configurations should be common solutions of the forward kinematics problems of both working modes. Solutions that are common to a couple of vanishing sets are solutions of the intersection of the sets, i.e. a solution is common to a couple of ideals if it satisfies all the homogeneous polynomial equations collected in their zero sets.

It is not difficult to figure out what is the number of feasible solutions of each intersection. To do that, Groebner bases are computed of each ideal union. It should be remarked that for this task both Study's parameters defining robot configuration and actuation displacements $q_{1,1}\ q_{1,2}\ q_{1,3}$ .are unknown. For this reason the computation of ideals bases is performed on the polynomial ring defined on the complex field by the lexicographic variables ordering $x_0 \succ x_1 \succ x_2 \succ x_3 \succ y_0 \succ y_1 \succ y_2 \succ y_3 \succ \delta_1 \succ \delta_2 \succ \delta_3$. The number of solutions of each ideal intersection is equal to the dimension of the vanishing set of the respective basis; thus, computation of such dimensions, whose results are reported in table 1, directly provides the dimension of the space of solutions for each operational modes intersection.

	$\mathcal{I}_{A,1}$	$\mathcal{I}_{A,2}$	$\mathcal{I}_{A,3}$	$\mathcal{I}_{A,4}$	$\mathcal{I}_{A,5}$	$\mathcal{I}_{A,6}$	$\mathcal{I}_{A,7}$	$\mathcal{I}_{A,8}$	$\mathcal{I}_{A,}$	$\mathcal{I}_{A,}$	$\mathcal{I}_{A,}$	$\mathcal{I}_{A,1}$	$\mathcal{I}_{A,1}$	$\mathcal{I}_{A,1}$	$\mathcal{I}_{A,1}$
$\mathcal{I}_{A,}$	3	2	2	2	2	1	1	1	1	1	1	0	0	0	0
$\mathcal{I}_{A,}$	2	3	1	1	1	2	0	2	0	2	0	1	-1	1	1
$\mathcal{I}_{A,}$	2	1	3	1	1	0	2	0	2	2	0	-1	1	1	1
$\mathcal{I}_{A,}$	2	1	1	3	1	0	2	2	0	0	2	1	1	1	-1
$\mathcal{I}_{A,}$	2	1	1	1	3	2	0	0	2	0	2	1	1	-1	1
$\mathcal{I}_{A,}$	1	2	0	0	2	3	-1	1	1	1	1	2	-1	-1	2
$\mathcal{I}_{A,}$	1	0	2	2	0	-1	3	1	1	1	1	-1	2	2	-1
$\mathcal{I}_{A,}$	1	2	0	2	0	1	1	3	-1	1	1	2	-1	2	-1
$\mathcal{I}_{A,}$	1	0	2	0	2	1	1	-1	3	1	1	-1	2	-1	2
$\mathcal{I}_{A,}$	1	2	2	0	0	1	1	1	1	3	-1	-1	-1	2	2
$\mathcal{I}_{A,}$	1	0	0	2	2	1	1	1	1	-1	3	2	2	-1	-1
$\mathcal{I}_{A,}$	0	1	-1	1	1	2	-1	2	-1	-1	2	3	-1	-1	-1
$\mathcal{I}_{A,}$	0	-1	1	1	1	-1	2	-1	2	-1	2	-1	3	-1	-1
$\mathcal{I}_{A,}$	0	1	1	1	-1	-1	2	2	-1	2	-1	-1	-1	3	-1
$\mathcal{I}_{A,}$	0	1	1	-1	1	2	-1	-1	2	2	-1	-1	-1	-1	3

Table 1: varieties intersection dimensions for each pair of working modes relative to joints configuration A.

Observation of the results collected in table 1 gives information about the conditions that must be respected for the transition between two working modes to take place:

- The intersection of vanishing sets having no common solutions has a dimension equal to -1; it means that no configuration is suitable for transition between mobilities.
- Dimension equal to 0 indicates that a finite number of solutions is available; therefore, the switch is possible in a restricted number (explicitly computable) of configurations.
- A dimension greater than 0 denotes the existence of an infinite number of solutions; as shown in the following for some cases, it is in general possible to extract a relation between actuation parameters that must be satisfied to make the transition possible.

Values within first row of table 1 arise a particular interest because they represent the transitions involving the pure rotational working mode: ideal denoting such mobility shares solutions with all other ideals. Therefore, an intermediate passage through this working mode makes possible, although not directly, all the transitions within table 1, even if some of the dimensions are equal to -1. Therefore, a deeper investigation on these specific cases looks reasonable.

- Intersections with solutions of hybrid working modes have dimension 2, thus the transition conditions are expressed by surfaces in the space of actuation parameters. The respective Groebner bases contain, within others, also polynomials whose vanishing sets define 4 planes π_i in the space of actuation displacements.

- Since the dimension of intersection with planar under actuated working modes ideals is1, it is expected that the respective spaces are curves. Indeed, 6 equations corresponding to many lines ρ_i are found.
- Dimension 0 denotes a finite number of solutions for intersections with pure translational working modes: indeed, points ε_i equations are present within generators of ideal intersection bases.

As mentioned, a switch between two kinds of mobility takes place through an appropriate reconfiguration of robot limbs that is possible only on singularity points. A simple confirmation of this fact is given by the vanishing of (3) on transition configurations.
The Jacobian analysis (see figure 3) pointed out that no singular points are present within the tetrahedral space enclosed into the singularity surface. Hence it is reasonable to deduce that singularities denoted by switching conditions coincide with the points of workspace borders represented in figure 3.

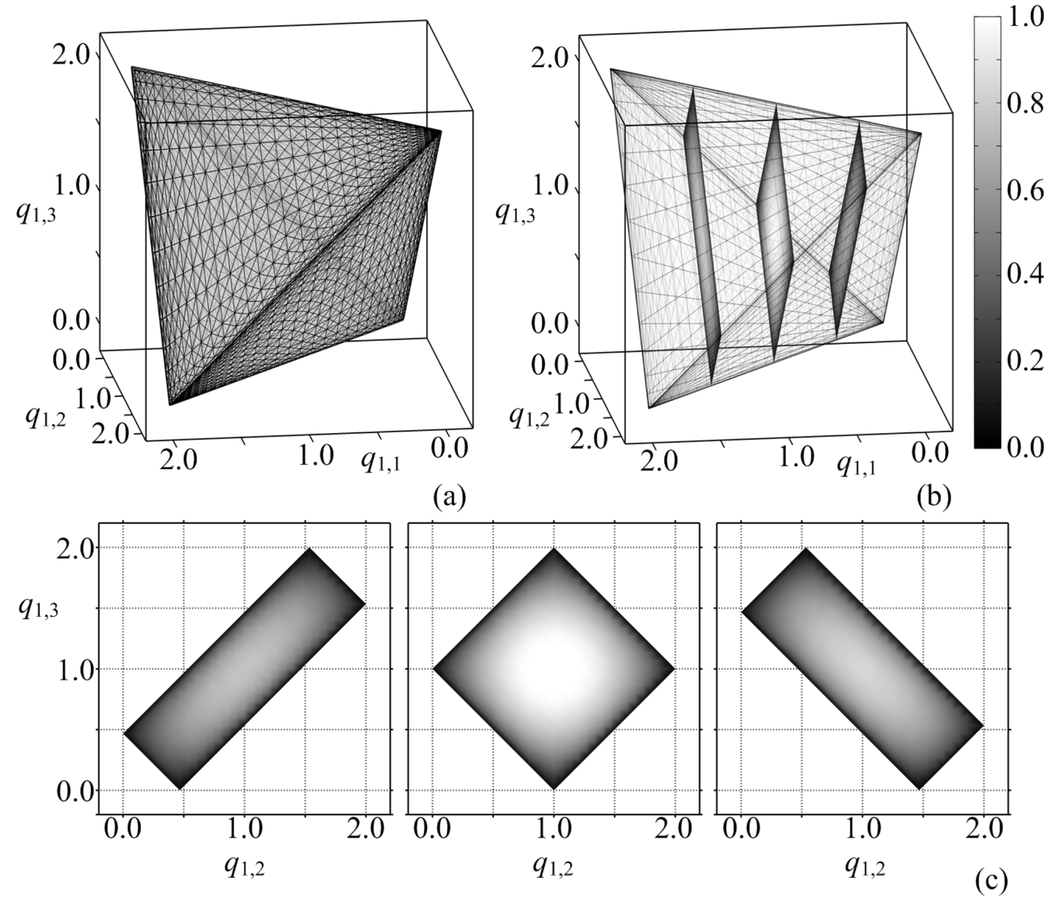

Figure 3: singularity surface according to jacobian analysis (a) and behaviour of determinant of Jacobian matrix on three different sections of robot actuated joints workspace (b), better detailed in (c). Data are computed for geometry parameters e = 1[m] and c = 1[m]; sections refer to $q_{1,1}$ = 0:5[m], $q_{1,1}$ = 1:0[m] and
$q_{1,1}$ = 1:5[m].

Figure 4 shows that switching conditions on actuated displacements actually have a correspondence on singularity surface, i.e. the 4 planes, the 6 lines and the 4 points respectively correspond to the faces, the edges and the vertices of the tetrahedral workspace. Therefore, a mathematical explicit description of singularity surfaces in terms of actuated joints displacements is given.

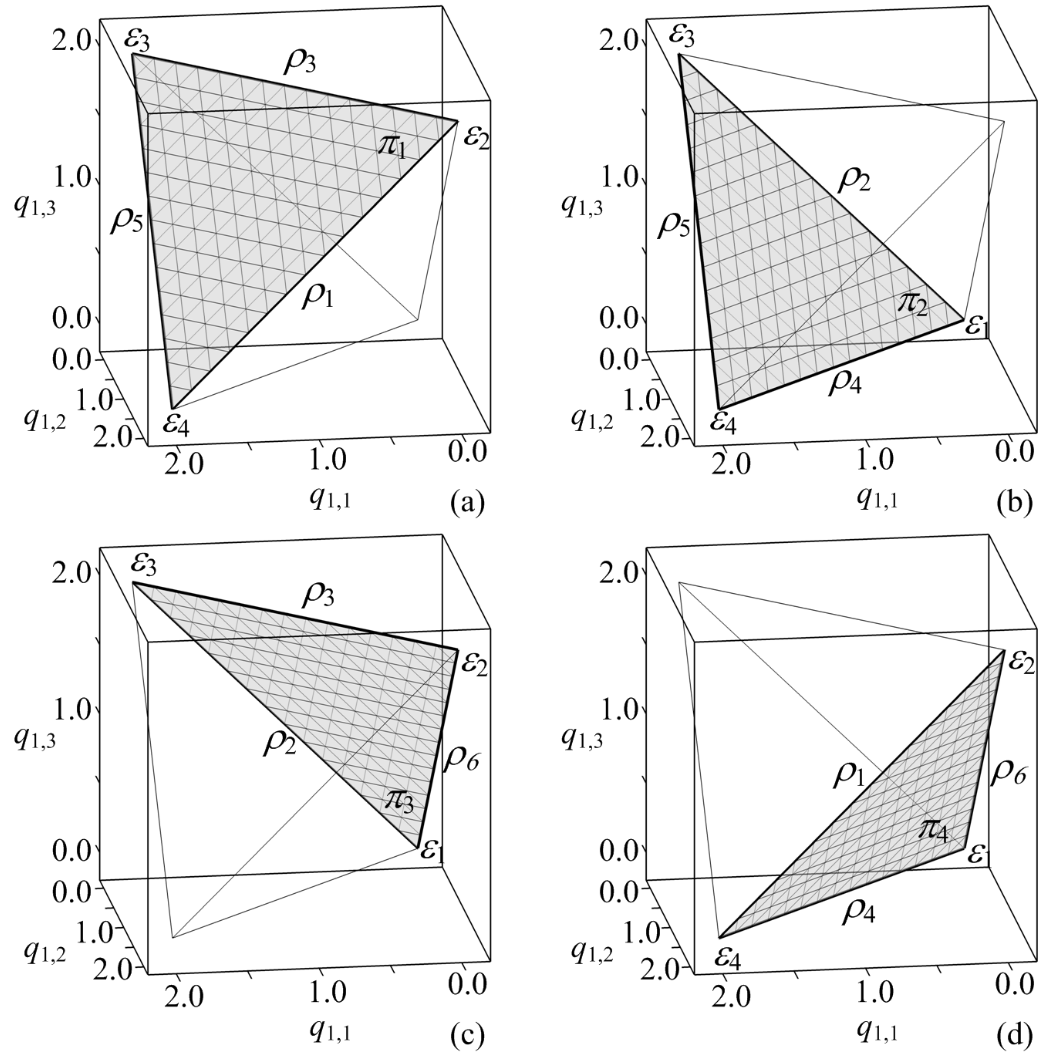

Figure 4: transition conditions in the space of actuated joints as defined in mobility analysis (b). Data are computed for geometry parameters e = 1[m] and c = 1[m].

The exact mathematical definition of singularity surface allows the definition of a convex region in the actuators displacements space which is devoid of singular configurations. Such information allows controlling the robot in a safe region avoiding problems deriving from the intersection of singular points. It is straightforward that the simplest convex region is a sphere centered in the homing position. The maximum radius that such sphere can assume is given by the distance of the center to the singularity surface, in particular to one of the four planes defining the tetrahedral workspace.

4 Conclusions

The kinematic analysis of the kinematotropic 3-CPU manipulator has been shown by means of Study's quadric representation. Such parametrization allowed to obtain in polynomial form all the equations needed to the formulation of the forward kinematics problem, thus allowing to use of an algebraic approach to the constraints analysis. The polynomial ideal collecting the actuation independent equations has been decomposed in several sub-ideals, each one characteristic of a particular operational mode of the 3-CPU manipulator. With the aim of seeking for transition conditions between various kinds of mobility, the spaces of common solutions of each pair of operational mode has been studied. To do that, a Groebner basis for the union of each sub-ideal pair has been computed. The dimensions of the respective vanishing sets gave the dimensions of the spaces of common solutions, if achievable, in the unknown actuation variables. This allowed to conclude that a transition path between various working modes is always possible due to the fact that pure rotational working mode shares solutions with every other mode. Moreover, observation of bases generators provided a precise formulation of the spaces of common solutions for transitions between pure rotational behaviour and other mobilities.
Afterwards, workspace analysis of the spherical 3-CPU parallel robot has been performed. The forward position kinematics as well as the differential kinematics of the manipulator have been faced in order to provide the needed analysis tools. A solution for the FKP is given by means of algebraic geometry and in particular exploiting again the Groebner basis formulation of the system of constraint equations. Screw theory has been then used to formulate the Jacobian matrix of the manipulator. The singularity surfaces have been traced within a limited active joints workspace by the zeroing of the determinant of this matrix. Exploiting the definition of transition configurations and the results achieved by the mobility analysis of the manipulator, a precise mathematical definition of workspace borders is given.

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